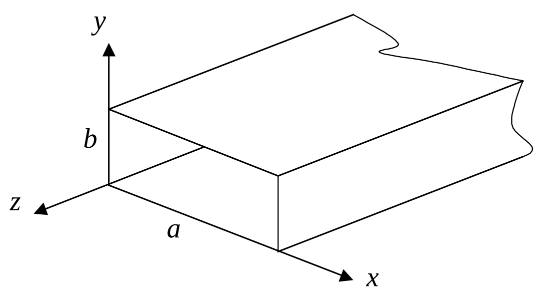
Waveguides (1)

Waveguides are an efficient means of transmitting microwaves. They can be hollow or filled with dielectric or other material. The cross section can be of any shape, but rectangular and circular are most common. First, we examine propagation in a rectangular waveguide of dimension *a* by *b*.



Waves propagate in the $\pm z$ direction: $\vec{E}(z),\ \vec{H}(z) \sim e^{\pm j\beta z}$. First separate Maxwell's equations into cartesian components (μ, ε refer to the material inside of the waveguide)

$$\left.\begin{aligned}
&\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \\
&-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \\
&\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z
\end{aligned}\right\} \nabla \times \vec{E} = -j\omega\mu\vec{H}$$

Waveguides (2)

$$\left.\begin{aligned}
&\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x \\
&-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\
&\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z
\end{aligned}\right\} \nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

Rearranging

$$E_x = \frac{-j}{\omega^2\mu\varepsilon - \beta^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{\omega^2\mu\varepsilon - \beta^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)$$

$$H_x = \frac{j}{\omega^2\mu\varepsilon - \beta^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{\omega^2\mu\varepsilon - \beta^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)$$

Waveguides (3)

The wave equations are:

$$\nabla^2 \vec{E} = -\omega^2 \mu\varepsilon \vec{E}$$
$$\nabla^2 \vec{H} = -\omega^2 \mu\varepsilon \vec{H}$$

Note that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial z^2} = (-j\beta)^2 = -\beta^2$ and the wave equations for the z components of the fields are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z = \left(\beta^2 - \omega^2 \mu\varepsilon\right) E_z$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z = \left(\beta^2 - \omega^2 \mu\varepsilon\right) H_z$$

<u>TEM waves do not exist in hollow rectangular waveguides</u>. The wave equations must be solved subject to the boundary conditions at the waveguide walls. We consider two types of solutions for the wave equations: (1) transverse electric (TE) and (2) transverse magnetic (TM).

Waveguides (4)

Transverse magnetic (TM) waves: $H_z = 0$ and thus $\vec{H}$ is transverse to the z axis. All field components can be determined from E_z. The general solution to the wave equation is

$$E_z(x,y,z) = E_z(x,y)e^{\pm j\beta z} = E_z(x)E_z(y)e^{\pm j\beta z}$$
$$= \left(A\cos(\beta_x x) + B\sin(\beta_x x)\right)\left(C\cos(\beta_y y) + D\sin(\beta_y y)\right)e^{\pm j\beta z}$$

where A, B, C, and D are constants. The boundary conditions must be satisfied:

$$E_z = 0 \text{ at } \begin{cases} x = 0 \rightarrow A = 0 \\ y = 0 \rightarrow C = 0 \end{cases}$$

Choose β_x and β_y to satisfy the remaining conditions.

$$E_z = 0 \text{ at } x = a: \quad \sin(\beta_x a) = 0 \quad \Rightarrow \quad \beta_x a = m\pi \quad \Rightarrow \quad \beta_x = \frac{m\pi}{a} \quad (m = 1,2,\ldots)$$

$$E_z = 0 \text{ at } y = b: \quad \sin(\beta_x b) = 0 \quad \Rightarrow \quad \beta_y b = n\pi \quad \Rightarrow \quad \beta_y = \frac{n\pi}{b} \quad (n = 1,2,\ldots)$$

Waveguides (5)

For TM waves the longitudinal component of the electric field for a +z traveling wave is given by

$$E_z(x,y,z) = U \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

where the product of the constants *AB* has been replaced by a new constant U. Each solution (i.e., combination of *m* and *n*) is called a mode. Now insert E_z back in the wave equation to obtain a separation equation:

$$\beta^2 = \omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

If $\beta^2 > 0$ then propagation occurs; $\beta^2 = 0$ defines a cuttoff frequency, $f_{c_{mn}}$,

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Waves whose frequencies are above the cutoff frequency for a mode will propagate, but those below the cutoff frequency are attenuated.

Waveguides (6)

Transverse electric (TE) waves: $E_z = 0$ and thus $\vec{E}$ is transverse to the z axis. All field components can be determined from H_z. The general solution to the wave equation is

$$\begin{aligned} H_z(x,y,z) &= H_z(x,y)e^{\pm j\beta z} = H_z(x)H_z(y)e^{\pm j\beta z} \\ &= \left(A\cos(\beta_x x) + B\sin(\beta_x x)\right)\left(C\cos(\beta_y y) + D\sin(\beta_y y)\right)e^{\pm j\beta z} \end{aligned}$$

But, from Maxwell's equations, $E_x \propto \dfrac{\partial H_z}{\partial y} \sim \cos\left(\dfrac{n\pi}{b}y\right)$ and $E_y \propto \dfrac{\partial H_z}{\partial x} \sim \cos\left(\dfrac{m\pi}{a}x\right)$.

Boundary conditions:

$$\begin{aligned} &E_x = 0 \text{ at } y = 0 \rightarrow D = 0 \\ &E_y = 0 \text{ at } x = 0 \rightarrow B = 0 \\ &E_x = 0 \text{ at } y = b \rightarrow \beta_y = \frac{n\pi}{b}, \quad n = 0,1,\ldots \\ &E_y = 0 \text{ at } x = a \rightarrow \beta_x = \frac{m\pi}{a}, \quad m = 0,1,\ldots \end{aligned}$$

Therefore,

$$H_z(x,y,z) = V\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z} \quad (m = n = 0 \text{ not allowed})$$

The same equation for cutoff frequency holds for both TE and TM waves.

Waveguides (7)

Other important relationships:

- Phase velocity for mode (m,n), $u_p = \dfrac{u}{\sqrt{1-\left(f_{c_{mn}} / f\right)^2}}$ where $u = 1/\sqrt{\mu\varepsilon}$ is the phase velocity in an unbounded medium of the material which fills the waveguide. Note the the phase velocity in the waveguide is larger than in the unbounded medium (and can be greater than c).
- Group velocity for mode (m,n), $u_g = u\sqrt{1-\left(f_{c_{mn}} / f\right)^2}$. This is the velocity of energy (information) transport and is less than the velocity in the unbounded medium.
- Wave impedance for mode (m,n),

$$Z_{\mathrm{TE}_{mn}} = \frac{\eta}{\sqrt{1-\left(f_{c_{mn}} / f\right)^2}}$$

$$Z_{\mathrm{TM}_{mn}} = \eta\sqrt{1-\left(f_{c_{mn}} / f\right)^2}$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the wave impedance in the unbounded medium.

- Phase constant for mode (m,n), $\beta_{mn} = \dfrac{\omega}{u_p} = \dfrac{\omega}{u}\sqrt{1-\left(f_{c_{mn}} / f\right)^2}$

Waveguides (8)

- Guide wavelength for mode (m,n), $\lambda_{g_{mn}} = \dfrac{\lambda}{\sqrt{1-\left(f_{c_{mn}}/f\right)^2}}$ where λ is the wavelength in the unbounded medium.

The dominant mode is the one with the lowest cutoff frequency. For rectangular waveguides with $a > b$ the TE_{10} mode is dominant. If a mode shares a cutoff frequency with another mode(s), then it is degenerate. For example, TE_{11} and TM_{11} are degenerate modes.

Example: If the following field exists in a rectangular waveguide what mode is propagating?

$$E_z = 5\sin\left(\frac{2\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)e^{-j2z}$$

Since $E_z \neq 0$ it must be a TM mode. Compare it with the general form of a TM mode field and deduce that m=2 and n=1. Therefore, it is the TM_{21} mode.

Waveguides (9)

Example: What is the lowest frequency that will readily propagate through a tunnel with a rectangular cross section of dimension 10m by 5m?

If the walls are good conductors, we can consider the tunnel to be a waveguide. The lowest frequency will be that of the dominant mode, which is the TE_{10} mode. Assume that the tunnel is filled with air

$$f_{c_{10}} = \frac{1}{2\sqrt{\mu_o \varepsilon_o}}\left(\frac{1}{a}\right) = \frac{c}{2(10)} = 15 \text{ MHz}$$

Example: Find the five lowest cutoff frequencies for an air-filled waveguide with a=2.29 cm and b=1.02 cm.

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{0.029}\right)^2 + \left(\frac{n}{0.0102}\right)^2}$$

Use Matlab to generate cutoff frequencies by looping through m and n. Choose the five lowest. Note that when both $m,n > 1$ then both TE and TM modes must be listed. (The frequencies are listed in GHz.)

$$\mathrm{TE}_{01}(14.71), \mathrm{TE}_{10}(6.55), \mathrm{TE}_{11} \text{ and } \mathrm{TM}_{11}(16.10), \mathrm{TE}_{20}(13.10)$$

Waveguides (10)

Example: Find the field parameters for a TE_{10} mode, f=10 GHz, a=1.5 cm, b=0.6 cm, filled with dielectric, $\varepsilon_r = 2.25$.

Phase velocity in the unbounded medium, $u = c/\sqrt{2.25} = 3\times10^8/1.5 = 2\times10^8$ m/s

Wavelength in the unbounded medium, $\lambda = u/f = 2\times10^8/1\times10^{10} = 0.02$ m

Cutoff frequency, $f_{c_{10}} = u/(2a) = \dfrac{c/\sqrt{2.25}}{(2)(0.015)} = 0.67\times10^{10}$ Hz

Phase constant, $\beta_{10} = \dfrac{\omega}{u}\sqrt{1-\left(f_{c_{mn}}/f\right)^2} = \dfrac{2\pi f}{c/\sqrt{2.25}}\underbrace{\sqrt{1-(0.067/1)^2}}_{0.745} = 74.5\pi$ radians

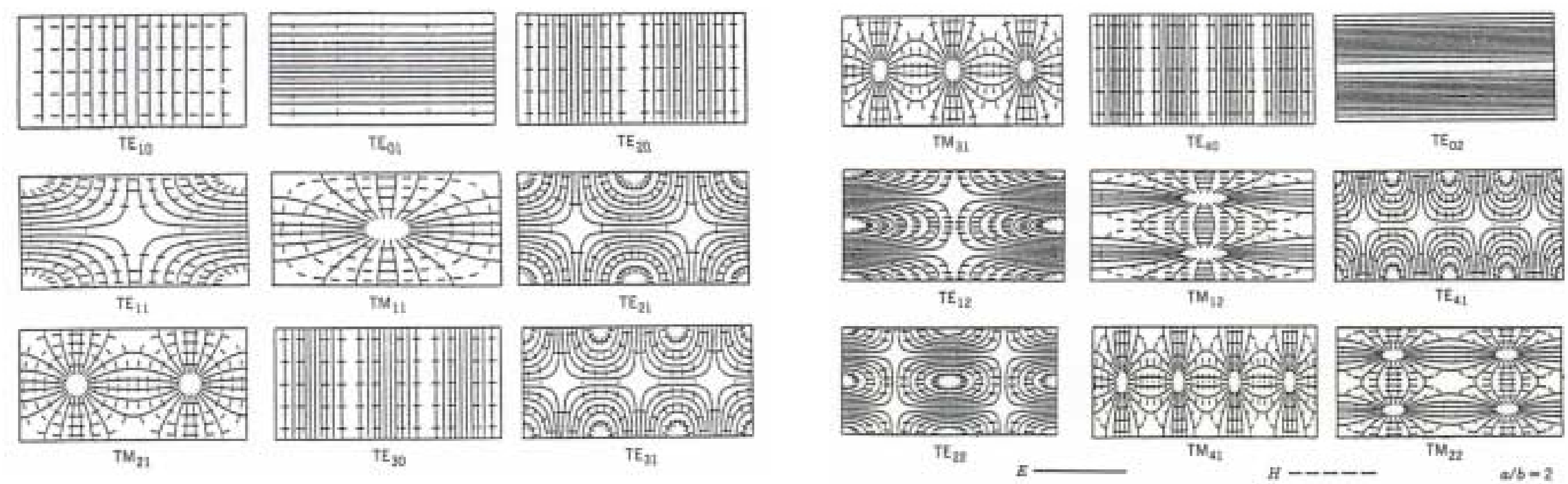
Guide wavelength, $\lambda_g = \dfrac{\lambda}{\sqrt{1-\left(f_{c_{mn}}/f\right)^2}} = \dfrac{0.02}{0.745} = 0.0268$ m

Phase velocity, $u_p = u/0.745 = 2\times10^8/0.745 = 2.68\times10^8$ m/s

Wave impedance, $Z_{\mathrm{TE}_{10}} = \dfrac{\eta}{\sqrt{1-\left(f_{c_{mn}}/f\right)^2}} = \dfrac{\eta_o/\sqrt{2.25}}{0.745} = \dfrac{(377)}{(0.745)(1.5)} = 337.4$ ohms

Group velocity, $u_g = 0.745u = (2\times10^8)(0.745) = 1.49\times10^8$ m/s

Mode Patterns in Rectangular Waveguide



From C.S. Lee, S. W. Lee, and L. L. Chuang, “Plot of Modal Field Distribution in Rectangular and Circular Waveguides,” *IEEE Trans. on MTT*, 1985.

Table of Waveguide Formulas

QUANTITY	TEM ($E_z = H_z = 0$)	TM ($H_z = 0$)	TE ($E_z = 0$)
WAVE IMPEDANCE, Z	$Z_{\text{TEM}} = \eta = \sqrt{\dfrac{\mu}{\varepsilon}}$	GENERAL: $Z_{\text{TM}} = \dfrac{\gamma}{j\omega\varepsilon}$ $f > f_c$: $\eta\sqrt{1-(f_c/f)^2}$ $f < f_c$: $\dfrac{-jh}{\omega\varepsilon}\sqrt{1-(f/f_c)^2}$	GENERAL: $Z_{\text{TE}} = \dfrac{j\omega\mu}{\gamma}$ $f > f_c$: $\dfrac{\eta}{\sqrt{1-(f_c/f)^2}}$ $f < f_c$: $\dfrac{j\omega\mu}{h\sqrt{1-(f/f_c)^2}}$
PROPAGATION CONSTANT, γ	$jk = j\omega\sqrt{\mu\varepsilon}$	GENERAL: $h\sqrt{1-(f_c/f)^2}$ $f > f_c$: $j\beta = jk\sqrt{1-(f_c/f)^2}$ $f < f_c$: $\alpha = h\sqrt{1-(f/f_c)^2}$	GENERAL: $h\sqrt{1-(f_c/f)^2}$ $f > f_c$: $j\beta = jk\sqrt{1-(f_c/f)^2}$ $f < f_c$: $\alpha = h\sqrt{1-(f/f_c)^2}$
PHASE VELOCITY, u_p	$u = \dfrac{1}{\sqrt{\mu\varepsilon}}$	GENERAL: ω/β $f > f_c$: $\dfrac{u}{\sqrt{1-(f_c/f)^2}}$ $f < f_c$: NO PROPAGATION	GENERAL: ω/β $f > f_c$: $\dfrac{u}{\sqrt{1-(f_c/f)^2}}$ $f < f_c$: NO PROPAGATION
VECTOR FIELD RELATIONSHIP	$\vec{H} = \dfrac{1}{Z_{\text{TEM}}}\hat{k}\times\vec{E}$	$\vec{E}_T = -\dfrac{\gamma}{h^2}\nabla_T E_z$ $\vec{H} = \dfrac{1}{Z_{\text{TM}}}\hat{z}\times\vec{E}$	$\vec{H}_T = -\dfrac{\gamma}{h^2}\nabla_T H_z$ $\vec{E} = -Z_{\text{TE}}\hat{z}\times\vec{H}$

Cutoff frequency: $f_c = \dfrac{h}{2\pi\sqrt{\mu\varepsilon}}$ Propagation constant: $\gamma = \sqrt{h^2 - k^2}$ Transverse Laplacian: $\nabla_T^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

For a rectangular waveguide (a by b): $h = \sqrt{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2}$ Guide wavelength: $\lambda_g = \dfrac{\lambda}{\sqrt{1-(f_c/f)^2}}$